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Parametric modelling of turbulence

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Some steps are taken towards a parametric statistical model for the velocity and velocity derivative fields in stationary turbulence, building on the background of existing theoretical and empirical knowledge of such fields. While the ultimate goal is a model for the three-dimensional velocity components, and hence for the corresponding velocity derivatives, we concentrate here on the streamwise velocity component. Discrete and continuous time stochastic processes of the first-order autoregressive type and with one-dimensional marginals having log–linear tails are constructed and compared with two large data-sets. It turns out that a first-order autoregression that fits the local correlation structure well is not capable of describing the correlations over longer ranges. A good fit locally as well as at longer ranges is achieved by using a process that is the sum of two independent autoregressions. We study this type of model in some detail. We also consider a model derived from the above-mentioned autoregressions and with dependence structure on the borderline to long-range dependence. This model is obtained by means of a general method for construction of processes with long-range dependence. Some suggestions for future empirical and theoretical work are given.

1. Introduction

The study of turbulence is a field of great (long-standing as well as current) theoretical and applied interest. An extensive view of very recent work is provided by the two Proceedings volumes Comte-Bellot & Mathieu (1987) and Fernholz & Fiedler (1989).

The developing theory of chaos and of fractals has, in recent years, thrown interesting light on aspects of turbulence (see, for example, Mandelbrot 1976, 1982; Eckmann & Ruelle 1985; Jones *et al.* 1988), whereas there has been relatively little progress in the study of turbulence by the statistical approach. Most of the existing statistical treatments are based on second- and higher-order moment properties or quite general studies of questions concerning stochastic solutions of the Navier–Stokes equations, or both (see, for example, Monin & Yaglom 1975; Vishik *et al.* 1979; Tatsumi *et al.* 1986), rather than on integrated, parametric modelling by means of stochastic process theory. Some suggestions for an approach of the latter type were made in Barndorff-Nielsen (1979) and in the present paper we discuss some further steps in that direction.

We focus on stationary turbulence, and begin by reviewing existing theoretical and empirical knowledge, in §§2 and 3 respectively. This leads, in §4, to the formulation of some desiderata for an integrated parametric modelling, and the following sections represent an attempt to meet some of these desiderata.

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Construction of linear autoregressions with one-dimensional marginal distributions of hyperbolic shape is considered in §5 and properties of the lag-k differences of such processes are discussed. The hyperbolic cosine distribution turns out to be particularly amenable for the present purposes. Furthermore, a new type of process is discussed, which seems useful when the data show autocorrelations over long ranges decaying more slowly than the short-range autocorrelations. Specifically, a sum of two independent autoregressions is considered. It is demonstrated how to construct such a process with a given one-dimensional marginal distribution. Continuous time analogues of some of the results are discussed in §6.

There are some indications of long-range dependence in turbulence (see Nordin *et al.* 1972). Furthermore, the idea of self-similarity has played a considerable role in discussions of turbulence (see, for example, van Atta & Park 1972; Frisch *et al.* 1978). Consequently, in §7 we review briefly the key concepts of stochastic processes with long-range dependence and self-similarity and introduce a general method for constructing such processes, while in §8 we apply that method to derive a process having characteristics of the kind seen in turbulent velocity fields.

The possible long-range properties of turbulence are in a sense disjoint from the classical theories of turbulence described in §2. The latter are concerned with the 'inertial subrange', i.e. with fluctuations on a very small scale where ideas of local isotropy can be used, whereas the long-range properties are for large fluctuations where the geometry of the flow comes into play.

In §9 we return to the main data sets reviewed in §3 and discuss these in the light of the models developed in §§5–8.

The final §10 consists of concluding remarks and some suggestions for further experimental and theoretical work.

2. Review of standard models for locally isotropic turbulence

It was L. F. Richardson who intuitively formulated the idea of locally isotropic turbulence through his famous poem: 'Big whorls have little whorls, which feed on their velocity; and little whorls have lesser whorls, and so on to viscosity' (Richardson 1922). When the Reynolds number $Re = UL/\nu$, where U and L are the characteristic velocity and length scales of the overall velocity field and ν is the viscosity of the fluid, is below a critical value Re_{cr} the flow is non-turbulent. Increasing Re to a value above the critical one, there will appear large-scale fluctuations drawing their energy from the mean motion. If the Reynolds number is sufficiently large the large-scale fluctuations will be unstable, i.e. they will break down and generate second-order disturbances of a smaller scale. This process will continue such that the energy from the mean motion is transferred to fluctuations of a smaller and smaller scale. The process continues until scales and characteristic velocity disturbances are reached for which the corresponding Reynolds number is of the order Re_{cr}, and the motion therefore is fluid-dynamically stable. At the smallest scales we have the largest values of the local velocity gradients and the kinetic energy is then lost by dissipation into heat. During this cascade process information on the geometry and structure of the mean motion is lost and the fluctuations become locally homogeneous and isotropic.

Kolmogorov formulated this mathematically by putting forward the hypothesis that the multidimensional probability distributions for the relative velocities

$u(x_0+r, t_0+\tau)-u(x_0,t_0)$ (for a collection of values of r and τ) are defined uniquely by $\bar{\epsilon}$ and ν for $r \ll L$ and $\tau \ll L/U$, where $\bar{\epsilon}$ is the mean energy dissipation and ν is the viscosity, which determines at what scales the kinetic energy of the flow is dissipated into heat. If we let $\eta = (\nu^3/\bar{\epsilon})^{\frac{1}{4}}$ this is a length scale, which indicates the size of the eddies where the energy dissipation takes place. A corresponding timescale is $\tau_\eta = (\nu/\bar{\epsilon})^{\frac{1}{2}}$. Kolmogorov also hypothesized that in the range $\eta \ll r \ll L, \tau_\eta \ll \tau \ll L/U$, which is called the inertial subrange, the relative velocities depend solely on $\bar{\epsilon}$. Monin & Yaglom (1975, §21.4) show by a dimensional argument that Kolmogorov's two hypotheses imply

$$E(\Delta_r u)^n = c_n(\bar{\epsilon}r)^{\frac{1}{3}n}, \quad \eta \ll r \ll L, \tag{2.1}$$

where $\Delta_r u = u(x+r,t)-u(x,t)$. In particular, we have Kolmogorov's 'two-thirds law' for $E(\Delta_r u)^2$, which gives rise to the 'five-thirds law' for the power spectrum.

Experimental findings related to (2.1) have shown that the original Kolmogorov theory is not strictly tenable (van Atta & Park 1972; Barndorff-Nielsen 1979; Anselmet *et al.* 1984). In 1962 Kolmogorov refined the theory by considering the original predictions to hold in a conditional setting given the value of a local average

$$\epsilon_r(x,t) = \int 1(|h| \leqslant r)\, \epsilon(x+h,t)\,\mathrm{d}h$$

of the dissipation. In this way the intermittency of the turbulence is taken into account, i.e. the fact that the flow contains regions with high activity and regions with low activity. The formula (2.1) becomes

$$E((\Delta_r u)^n \mid \epsilon_r) = c_n(\epsilon_r r)^{\frac{1}{3}n}\tilde{\beta}_n[r/(\nu^3/\epsilon_r)^{\frac{1}{4}}] \tag{2.2}$$

with $\tilde{\beta}_n(x) \sim ax^{\frac{2}{3}n}$ for $x \to 0$ and $\tilde{\beta}_n(x) \to 1$ for $x \to \infty$, cf. Monin & Yaglom (1975, p. 353, pp. 592–593). The mean of ϵ_r is $\bar{\epsilon}$ and Kolmogorov (1962) furthermore assumed that $\ln \epsilon_r$, being a fluctuating quantity itself, is normally distributed with variance $A(x,t)+\mu \log_{10} L/r$, where μ is a universal constant. If we substitute 1 for $\tilde{\beta}_n$ in (2.2) and take mean values we then obtain the refined estimate

$$E(\Delta_r u)^n = b_n \bar{\epsilon}^{\frac{1}{3}n} r^{\frac{1}{3}n}(L/r)^{\frac{1}{18}\mu n(n-3)}. \tag{2.3}$$

Note, that the power $\frac{1}{3}n$ in formula (2.1) has been reduced to $\frac{1}{18}\mu n(n-3)$, where empirical studies have indicated a value of μ around 0.5. The log-normality assumption for ϵ_r has been criticized on both theoretical and empirical grounds (see, for example, Gibson & Masiello 1972).

As a prelude to the discussion below, of what is known as the β-model, we mention that if ϵ_r is assumed to be gamma distributed with variance $C\epsilon^{-2}(L/r)^\mu$, rather than log-normally distributed, then instead of (2.3) we obtain

$$\tilde{b}_n\, \epsilon^{-\frac{1}{3}n} r^{\frac{1}{3}n} \Gamma\{\tfrac{1}{3}n+[C(L/r)^\mu]^{-1}\}[C(L/r)^\mu]^{\frac{1}{3}n}/\Gamma\{[C(L/r)^\mu]^{-1}\}. \tag{2.4}$$

This expression is approximately $d_n \bar{\epsilon}^{\frac{1}{3}n} r^{\frac{1}{3}n}[C(L/r)^\mu]^{\frac{1}{3}(n-3)}$ for $C(L/r)^\mu$ large, i.e. for small values of r the dependence on r is $r^{\frac{1}{3}n-\frac{1}{3}\mu(n-3)}$.

We now turn to a different dynamical model, introduced in Frisch *et al.* (1978). The authors consider a decreasing sequence l_n of scales of eddies, and assume that only a fraction β_n of space is occupied by eddies of scale l_n. Assuming $\beta_n = (l_n/l_0)^m$, called the β-model, and assuming that the distributions of velocity differences within active regions of different scales become identical under an appropriate scaling, the authors derive the formula

$$E(\Delta_r u)^n = c_n\, \bar{\epsilon}^{\frac{1}{3}n} r^{\frac{1}{3}n} r^{-\frac{1}{3}\mu(n-3)}.$$

This deviates from (2.3) and is close to (2.4).

Summing up, the above approaches based on dimensional analysis and scaling arguments lead to relations for moments and correlation structure. A description in terms of stochastic processes, where also the statistical distributions are taken into account, does, however, not follow from these considerations. We stress here that the above discussion deals with the inertial subrange of turbulence, i.e. small-scale variations with frequencies typically less than 1 s, whereas also large scales are considered in the following.

3. Some main data-sets

An important and useful concept in turbulence is Taylor's frozen field hypothesis. For the streamwise velocity component $u(x,t)$ of a stationary turbulent field it specifies that for any location x and for any positive r the velocity difference

$$u(x+r,t)-u(x,t)$$

follows the same distribution as the time-wise difference $u(x,t+\Delta t)-u(x,t)$ where $r = v\Delta t$ and v is the mean speed. This hypothesis is used to convert spatial measurements to temporal measurements and vice versa. It lies behind the analyses of the two major data sets discussed below.

A review of the empirical evidence concerning the statistical properties of the streamwise velocity and velocity derivative processes in high Reynolds number turbulence, such as existed up till 1979, was given in Barndorff-Nielsen (1979). Referring to that paper for details we may roughly summarize the evidence as follows.

The one-dimensional marginal distributions of the velocity and velocity derivative processes are generally and to a remarkable degree of approximation of a hyperbolic shape; and very far from being gaussian laws. However, the velocity differences are such that for large time-wise or spatial distances their distribution approaches a parabolic, i.e. gaussian, shape, whereas for small such distances the distributional tails become heavier than the log–linear tails characteristic of the hyperbolic shape, i.e. in the logarithmic plotting the distributions 'skirt out', and the more so the smaller the distance. See in particular figure 10 of Barndorff-Nielsen (1979).

Later studies have confirmed these findings. In particular, the remarkable tendency for observed distributions of velocity differences to follow a hyperbolic shape have been further put in evidence in a variety of contexts, see for instance Dinkelacker *et al.* (1989) and Barndorff-Nielsen *et al.* (1989).

Concerning Kolmogorov's assumption of log–normality for the distribution of the averaged dissipation ϵ_r, as mentioned above, this has not been clearly established. Gibson & Masiello (1972) considered the distribution of ϵ_r for different values of r. Although a log–normal distribution is fitted it is clear from figure 3 of their paper that $\ln \epsilon_r$ is much more concentrated near the mean value than is a normal distribution (at least in the upper tail, the lower tail is more undetermined). This in turn invalidates formula (2.3) for high-order moments. In general a check of the Kolmogorov model in terms of (2.3) is very sensitive to the detailed description of the tails of the distribution of ϵ_r.

The log–normal model deviates from the β-model discussed in §2, and Anselmet *et al.* (1984) have made a very detailed experimental study of the higher order moments in order to distinguish between the log–normal model and the β-model. They conclude that the log–normal model fits the data best. However, this conclusion is based on fitting a linear relation between $\ln E(\Delta_r u)^n$ and $\ln r$, and looking at their figures a striking feature is that the curves are concave. So it seems that the log–normality assumption is not suitable and that the β-model, in its pure form, is too simplified. Interestingly, the moments (2.4) based on a gamma distribution will give rise to a concave curve. It should be noted here, however, that when evaluating the mean value of (2.2), leading to (2.3), it is not correct to replace $\tilde{\beta}_n$ by

$$\lim_{x\to\infty} \tilde{\beta}_n(x) = 1,$$

the more so the higher n is. Actually, if we believe in the log–normal model for ϵ_r the influence of $\tilde{\beta}_n$ will make the logarithm of the moments of $\Delta_r u$ a slightly concave function of $\ln r$.

We proceed to discuss a part of another extensive data-set recorded in late September 1985 on the beach at Ferring on the Danish west coast. (For more information on the data-set and for an extensive study of other aspects of the data see Mikkelsen (1988, 1989).) A 30 m mast was erected on the shore 50–70 m from the shoreline and turbulence was measured at the top with a sonic anemometer. All three velocity components were measured, but here we consider only the streamwise component u. The measuring system produced values averaged over 100 ms and the signal was sampled with a 10 Hz frequency. Three runs were made and for the first run (run 1), which we consider here, the mean wind velocity was 7.1 m s^{-1}. A calculation shows that with these data we can study turbulence scales ranging from the upper part of the inertial subrange and upwards. Thus the data will not directly throw light on Kolmogorov's hypotheses, but they are of interest for a description of other aspects of turbulence. In figure 1*a* (see §9) 500 data points of the streamwise velocity are shown. Figure 1*b* shows the corresponding lag 1 differences $a_n = u_n - u_{n-1}$. A log-histogram of the marginal distribution of the lag 1 difference reveals log–linear tails and a density of hyperbolic shape, see figure 2.

We return to these data in §9.

4. Desiderata for a parametric modelling

The experimental and theoretical evidence concerning the statistical properties of the velocity and velocity derivative processes u and a in the mean wind direction of a stationary, high Reynolds number, turbulent wind field – as discussed above – may be summarized as follows.

1. The velocity and velocity derivative processes u and a can be considered as stationary stochastic processes.

2. The distribution of $\Delta_r u$ is, for not too small r, of the hyperbolic type with both mean and skewness close to 0. For small r the distributions have heavier tails than the hyperbolic distributions, the more so the smaller r is.

3. The correlation function $\rho_u(r)$ of the velocity process is approximately of the form

$$\rho_u(r) = 1 - \text{const.} \times r^{\frac{2}{3}}$$

for r in the inertial subrange.

4. The energy dissipation process $\epsilon_r(t)$ defined by (2.2) has a one-dimensional marginal distribution whose upper tail is lighter than that of a log–normal distribution. This contradicts Kolmogorov's modified theory.

5. The velocity process shows autocorrelation over long ranges, perhaps it is even close to being long-range dependent in the formal sense. For further discussion see the §§7, 8 and 9.

Our aim is now to construct stochastic processes u that, as far as possible, exhibit the above traits, and to compare the resulting models more closely with the main available data.

5. Linear autoregressive processes: hyperbolic cosine autoregressive (AR) process

It is possible to construct an AR(1) process having a given one-dimensional marginal distribution provided that distribution is self-decomposable; and only then (see Cox (1981), in particular the contribution of L. Bondesson to the discussion of that paper). In fact, if

$$x(n) = \rho x(n-1) + z(n), \quad |\rho| < 1, \tag{5.1}$$

where the innovations $z(n)$ are independent and identically distributed, then the characteristic functions C must satisfy

$$C(x;\zeta) = C(x;\rho\zeta)\, C_\rho(z;\zeta) \tag{5.2}$$

in an obvious notation, and this is the defining property of self-decomposability. Note the somewhat unsatisfactory feature that the distribution of the innovations generally depends on ρ. For present purposes this does not, however, seem of crucial importance.

Now, consider the lag k difference process corresponding to the autoregression (5.1):

$$a^{[k]}(n) = x(n) - x(n-k). \tag{5.3}$$

We have $$a^{[k]}(n) = -(1-\rho^k)\, x(n-k) + \rho^{k-1} z(n-k+1) + \ldots + z(n)$$

and hence, by (5.2),

$$\begin{aligned} C(a^{[k]};\zeta) &= C(x; -(1-\rho^k)\,\zeta)\, C_\rho(z;\rho^{k-1}\zeta) \ldots C_\rho(z;\zeta) \\ &= C(x; -(1-\rho^k)\,\zeta)\, C(x;\zeta)/C(x;\rho^k\zeta). \end{aligned} \tag{5.4}$$

This implies, in particular, that if the distribution of $x(n)$ is symmetric around 0 and if $\rho = 2^{-1/k}$ then $C(a^{[k]};\zeta) = C(x;\zeta)$, i.e. $a^{[k]}(n)$ and $x(n)$ have identical one-dimensional marginal distributions.

There are two important classes of one-dimensional distributions with shape of the hyperbolic type: the hyperbolic distributions (Barndorff-Nielsen 1977; Barndorff-Nielsen *et al.* 1985) and the generalized logistic distributions. As shown respectively by Halgreen (1979) and Barndorff-Nielsen *et al.* (1982) both these kinds of distributions are self-decomposable. For the present initial study we shall restrict discussion to the case where the distributions are symmetric generalized logistic with mean 0 and scale parameter equal to 1. This leaves a single parameter $\alpha > 0$, which may be thought of as expressing the kurtosis.

The density function and the characteristic function of the selected generalized logistic distribution are, respectively,

$$f(x) = B(\alpha, \alpha)^{-1} \exp(\alpha x)/\{1+\exp(x)\}^{2\alpha}, \tag{5.5}$$

where B denotes the beta function, and

$$C(x;\zeta) = B(\alpha+\mathrm{i}\zeta, \alpha-\mathrm{i}\zeta)/B(\alpha,\alpha). \tag{5.6}$$

The corresponding innovation process has characteristic function

$$C_\rho(z;\zeta) = B(\alpha+\mathrm{i}\zeta, \alpha-\mathrm{i}\zeta)/B(\alpha+\mathrm{i}\rho\zeta, \alpha-\mathrm{i}\rho\zeta). \tag{5.7}$$

When $\alpha = \frac{1}{2}$ the expression (5.6) simplifies to

$$[\cosh(\pi\zeta)]^{-1}, \tag{5.8}$$

the corresponding density function being

$$[\pi \cosh(\tfrac{1}{2}x)]^{-1}, \tag{5.9}$$

i.e. the distribution has the hyperbolic cosine form. This particular distribution has a number of analytic properties that are useful for our purposes and henceforth we limit our discussion to this distribution.

The characteristic function of the innovation process associated with (5.8) is

$$C_\rho(z;\zeta) = \cosh(\pi\rho\zeta)/\cosh(\pi\zeta). \tag{5.10}$$

We have

$$(\cosh x)^{-1} = 2\sum_{k=0}^{\infty}(-1)^k \exp[-(2k+1)|x|]. \tag{5.11}$$

Hence $C_\rho(z;\cdot)$ may be expanded as

$$C_\rho(z;\zeta) = \{\exp[-(1+\rho)\pi|\zeta|]+\exp[-(1-\rho)\pi|\zeta|]\}\sum_{k=0}^{\infty}(-1)^k \exp[-2k\pi|\zeta|] \tag{5.12}$$

and Fourier inversion then shows that the probability density function of the innovation variates is

$$f_\rho(z) = \sum_{k=0}^{\infty}(-1)^k[c(z;\pi(1+\rho+2k))+c(z;\pi(1-\rho+2k))] \tag{5.13}$$

where $c(z;\delta)$ denotes the density function of the Cauchy distribution with scale parameter δ, i.e.

$$c(z;\delta) = [\pi\delta(1+z^2/\delta^2)]^{-1}. \tag{5.14}$$

From (5.4) we find that the lag-one difference process $a(n)$ of the hyperbolic cosine autoregression has characteristic function

$$C_\rho(a;\zeta) = \cosh(\rho\pi\zeta)/\{\cosh(\pi\zeta)\cosh[(1-\rho)\pi\zeta]\} \tag{5.15}$$

and from this, and using (5.11) again, we obtain the probability density function of $a(n)$ as

$$f_\rho(a) = 2\Sigma(-1)^{k+j}[c(a;2\pi\{(1-\rho)(k+1)+j\})+c(a;2\pi\{(1-\rho)k+j\})], \quad (5.16)$$

where the summation is over all non-negative j and k.

For accordance with point 2 in our list of desiderata (§4) it is desirable that for large k the distribution of $a^{[k]}$ is close to normal while for $k < k_0$, where $k_0 = (\ln 2)/(-\ln\rho)$, the distribution 'skirts out', the more so the smaller is k. (Recall from above that if $k = k_0$ then $a^{[k]}$ has the same distribution as x.) We conjecture that the hyperbolic cosine process has this property. The fact that for $\rho\to 1$ we have $f_\rho(z) \sim c(z;\pi(1-\rho))$ (cf. formula (5.12)) supports this conjecture. To further substantiate the conjecture we now consider the fourth standardized cumulants γ_2 and $\gamma_2^{[k]}$ of the processes x and $a^{[k]}$. These are related by

$$\gamma_2^{[k]} = \kappa(k,\rho)\,\gamma_2, \quad (5.17)$$

where

$$\kappa(k,\rho) = \frac{(1-\rho^k)^4-1-\rho^{4k}}{[(1-\rho^k)^2+1-\rho^{2k}]^2}. \quad (5.18)$$

If k is large and ρ is close to 1, $\kappa(k,\rho)$ is small, which corresponds to an almost normal distribution. Precisely, $\kappa(k,\rho)\to 0$ for $k(1-\rho)\to\infty$, the latter requirement being equivalent to $k/k_0\to\infty$. For small k, for example $k=1$, $\kappa(k,\rho)\to\infty$ when $\rho\to 1$, indicating that the distribution of $a^{[1]}$ 'skirts out'.

For use in situations where there is correlation over long ranges decaying more slowly than the short-range correlation (such a situation will be discussed in §9), we introduce the following type of model. Let $x^{(1)}$ and $x^{(2)}$ be stationary autoregressions defined by

$$x^{(i)}(n) = \rho_i x^{(i)}(n-1)+z^{(i)}(n), \quad i = 1,2, \quad (5.19)$$

where $|\rho_1| < |\rho_2| < 1$ and $z^{(1)}$ and $z^{(2)}$ are two independent sequences of independent, identically distributed random variables. Set $\chi = \operatorname{var}(x^{(1)}(n))/\operatorname{var}(x^{(2)}(n))$. We define a process by

$$x(n) = x^{(1)}(n)+x^{(2)}(n). \quad (5.20)$$

The lag k autocorrelation of x is given by

$$\rho(k) = (\chi\rho_1^k+\rho_2^k)/(\chi+1). \quad (5.21)$$

For large values of k the correlation function $\rho(k)$ behaves like that of an AR(1) process with regression coefficient ρ_2.

In a similar way as for AR(1) processes we can construct a process x of the type (5.20) with a given one-dimensional marginal distribution if and only if that distribution is selfdecomposable. More specifically, let $C(\zeta)$ denote the characteristic function of the given distribution. A self-decomposable distribution is infinitely divisible and, moreover, so is $C(\zeta)/C(\rho\zeta)$ for $0<\rho<1$, see Feller (1971, p. 589). Hence $C(\zeta)^\delta$ as well as $C(\zeta)^\delta/C(\rho\zeta)^\delta$ are characteristic functions for $0<\delta<1$. In particular, $C(\zeta)^\delta$ defines a self-decomposable distribution. We can then construct independent AR(1) processes $x^{(1)}$ and $x^{(2)}$ such that their marginals have characteristic functions $C(\zeta)^\delta$ and $C(\zeta)^{1-\delta}$, respectively. In conclusion, the process x defined by (5.20) has marginals given by $C(\zeta)$ and $\chi = \delta/(1-\delta)$.

The above results about the lag k difference process for AR(1) processes carry immediately over to the new type of processes due to the independence and the linearity of the construction of x. In particular, (5.4) holds.

6. Continuous time analogues of autoregressive processes: Laplace diffusion

Continuous time analogues of the linear AR(1) process (5.1) are provided by certain diffusion processes defined by stochastic differential equations of the form

$$\mathrm{d}x(t) = -\beta x(t)\,\mathrm{d}t + b(x(t)\,;\beta)\,\mathrm{d}w(t), \tag{6.1}$$

where β is a regression parameter and $w(t)$ denotes the Wiener process.

Consider for a moment the general stochastic differential equation

$$\mathrm{d}x(t) = a(x(t))\,\mathrm{d}t + b(x(t))\,\mathrm{d}w(t), \tag{6.2}$$

and assume that it has a unique solution $x(t)$, which is a regular diffusion on the real line. Under regularity conditions, $x(t)$ is a stationary process whose one-dimensional distributions have probability density function given by

$$f(x) = c\exp\left[2\int_0^x \frac{a(y)}{b^2(y)}\,\mathrm{d}y\right]\Big/b^2(x), \tag{6.3}$$

where c is a constant to be chosen such that f is a probability density function, see Kent (1978). Sufficient regularity conditions are that $b(x) > 0$, that f is integrable, and that the diffusion is conservative. The last condition, which means that the probability mass is preserved, is ensured if

$$\int_0^\infty [F(x)-F(0)]\,s(x)\,\mathrm{d}x = \int_{-\infty}^0 [F(0)-F(x)]\,s(x)\,\mathrm{d}x = \infty, \tag{6.4}$$

where F is the distribution function corresponding to f and

$$s(x) = \exp\left\{-2\int_0^x \frac{a(y)}{b^2(y)}\,\mathrm{d}y\right\} \tag{6.5}$$

is the density function of the scale measure of $x(t)$.

As a particular example, which we require in §8, the equation

$$\mathrm{d}x(t) = -\beta x(t)\,\mathrm{d}t + \{2\beta(1+|x(t)|)\}^{\frac{1}{2}}\,\mathrm{d}w(t) \tag{6.6}$$

has a stationary solution with one-dimensional marginal distribution the standard Laplace distribution

$$f(x) = \tfrac{1}{2}\mathrm{e}^{-|x|}. \tag{6.7}$$

More generally, equation (6.2) has a stationary solution with a given marginal one-dimensional density $f(x) > 0$ provided the function $b(x(t))$ is chosen to satisfy

$$\tfrac{1}{2}b(x)^2 = f(x)^{-1}\left[\int_{-\infty}^x f(y)\,a(y)\,\mathrm{d}y + C\right], \tag{6.8}$$

where C is a constant. This is, of course, provided the integral exists.

7. Processes with long-range dependence and self-similarity

A stationary stochastic process $x(t)$ with mean 0 is said to exhibit long-range dependence if its correlation function $\rho(r) = E[x(t)\,x(t+r)]/E[x(t)]^2$ is asymptotically, as $r \to \infty$, of the form

$$\rho(r) \sim L(r)\, r^{-2H} \tag{7.1}$$

for some constant H with $0 < H < \frac{1}{2}, L(r)$ being a slowly varying function.

Closely related to the concept of long-range dependence is the idea of self-similarity. When $x(t)$ is a discrete time process with correlation function satisfying (7.1) then for every $n = 1, 2, \ldots$, the derived sum process

$$x_n(t) = x(nt+1) + \ldots + x(nt+n)$$

has exactly or approximately, for large r, the same correlation function as $x(t)$, and in this sense the processes $x_n(t), n = 1, 2, \ldots$, are, exactly or approximately, self-similar. The correlation functions are exactly equal if

$$\rho(r) = \tfrac{1}{2}\delta^2 r^{2(1-H)},$$

where δ denotes the central difference operator. The heuristic idea is that the cumulative sum process $s(t) = x(1) + \ldots + x(t)$ will exhibit the same correlation structure whether looked at 'close up' or 'from a smaller or larger distance'. For continuous time processes $s(t)$ is, correspondingly, defined as an integral and $s(t)$ is said to be exactly self-similar with exponent H provided that for any positive c the process $s_c(t) = s(ct)$ follows the same probabilistic law as the process $c^H s(t)$.

A review of the roles of long range dependence and self-similarity in statistics was given by Cox (1984). In that paper a method for construction of processes with long-range dependence by weighted integration of processes with short-range dependence was introduced, and this has recently been applied in a study of the relations of nonlinearity and time irreversibility to long range dependence (Cox 1990). A mathematically rigorous treatment of the Cox integral would seem to require the use of integration by means of random measures. No such treatment is presently available, and we shall instead propose a somewhat similar construction that does not meet with the same difficulties and which turns out to be very convenient for our purposes.

This alternative construction consists of considering a sequence of independent continuous time processes $x_k(t), k = 1, 2, \ldots$, having identical probability laws, and defining a new process $x(t)$ by

$$x(t) = \sum_{k=1}^{\infty} x_k(k^{-1}t)\, w(k), \tag{7.2}$$

where the $w(k)$ are certain weights, satisfying

$$\sum w(k)^2 < \infty. \tag{7.3}$$

The processes $x_k(\cdot)$ are assumed to be stationary and square integrable with mean 0. The process $x(\cdot)$ is then well-defined as an L^2 limit (for instance, Loève 1963, §9.4). We denote the common correlation function of the processes $x_k(\cdot)$ by $\gamma(r)$ and the correlation function of the weighted process $x(t)$ by $\rho(r)$.

Now suppose that for $k \to \infty$

$$w(k) \sim \text{const.} \times k^{-A}.$$

Here A denotes a constant that is assumed to be greater than $\frac{1}{2}$ in order that (7.3) be satisfied. Then, provided $\gamma(r)$ is a continuous function of r and $\gamma(r) \rightarrow 0$ as $r \rightarrow \infty$, (7.1) holds with $H = A - \frac{1}{2}$, and there is long range dependence if $\frac{1}{2} < A < 1$.

Recently Davison & Cox (1989) studied the limit distributions of sums of a special type of random variables having finite variance and long-range dependence, via cumulants and simulations. In particular, the simulations revealed a tail behaviour of the distributions close to log–linearity, as is the case for the hyperbolic distributional shape.

8. Quasi-long-range generalized logistic processes

As indicated previously, one of our interests in the present paper is in constructing stationary continuous time processes with long range or close to long-range dependence whose one-dimensional marginal distributions are of the hyperbolic shape, thus in particular having log–linear tails. We wish to use such a process as a model for the primary velocity component of a stationary turbulent wind field. A decisive test of the usefulness of such a model is whether the velocity differences under the model exhibit traits like those observed experimentally and discussed in §3.

With $x(t)$ defined by (7.2), the characteristic function of $x(t)$ satisfies

$$C(x(t);\zeta) = \prod_{k=1}^{\infty} C(x_1(0);w(k)\,\zeta), \tag{8.1}$$

where the stationarity of the processes has been taken into account. We wish to discuss the question of whether the processes $x_k(t)$ can be so defined that $x(t)$ follows a distribution of hyperbolic shape.

The characteristic function (5.6) of the symmetric generalized logistic distribution has, as noted in Barndorff-Nielsen *et al.* (1982), the infinite product representation

$$B(\alpha+\mathrm{i}\zeta, \alpha-\mathrm{i}\zeta)/B(\alpha,\alpha) = \prod_{k=1}^{\infty} \{1+\zeta^2/(\alpha+k-1)^2\}^{-1}. \tag{8.2}$$

Here the general factor $[1+\zeta^2/(\alpha+k-1)^2]^{-1}$ is the characteristic function of the Laplace distribution with density

$$\tfrac{1}{2}(\alpha+k-1)\exp[-(\alpha+k-1)|x|]. \tag{8.3}$$

This distribution is self-decomposable, since it is one of the limit distributions of the class of hyperbolic distributions. For each k, by (6.6), we can therefore define a stationary diffusion process $x_k(t)$ having the standard Laplace distribution (6.7) as the one-dimensional marginal distribution. Letting

$$w(k) = (a+k-1)^{-1}$$

the distribution of $x(t) = \Sigma\, x_k(tk^{-1})\, w(k)$ is then the generalized logistic distribution (5.5). Furthermore, we have $\Sigma\, w(k)^2 < \infty$ and

$$\rho(r) \sim \text{const.} \times r^{-1}, \tag{8.4}$$

corresponding to the border line of long-range dependence.

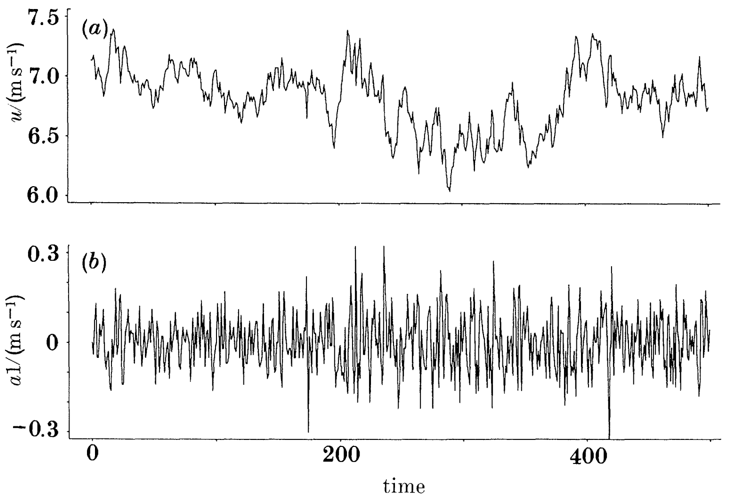

Figure 1. (*a*) Run 1, velocity; (*b*) run 1, lag-1 differences.

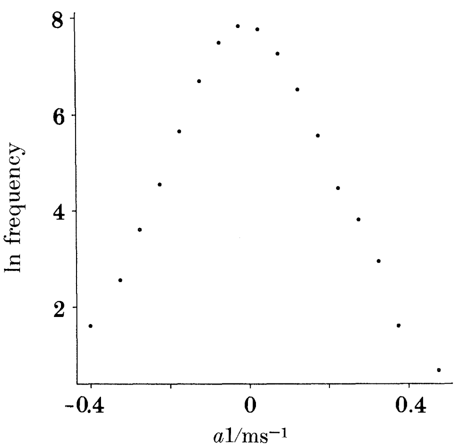

Figure 2. Run 1, distribution of lag-1 differences.

9. Return to data

For comparison with the data-sets discussed in Barndorff-Nielsen (1979), and summarized in §3 above, we have simulated the hyperbolic cosine autoregression introduced in §5 with a regression coefficient of 0.8. The resulting process and its lag-1 differences are given in figure 3, and figure 4 shows the one-dimensional log frequency distributions of the lag-1, lag-2 and lag-3 difference processes. As expected from the discussion in §5 these three distributions have tails that are heavier than log–linear, the more so the smaller the lag, and the lag-3 distribution is close to the hyperbolic cosine, in accordance with the fact that $\ln 2/-\ln \rho = 3.11$.

Figure 1*a* shows the variation with time of the streamwise velocity component u in run 1 from the Ferring data, and the corresponding lag-1 difference process is plotted in figure 1*b*. Figure 2 presents the log frequency distribution of this difference process, and one notes that the distribution is close to the symmetric hyperbolic shape, with a slight tendency to 'skirts'. Thinking of the velocity process as a hyperbolic cosine process and of the 'reproductivity' of linear autoregressions under

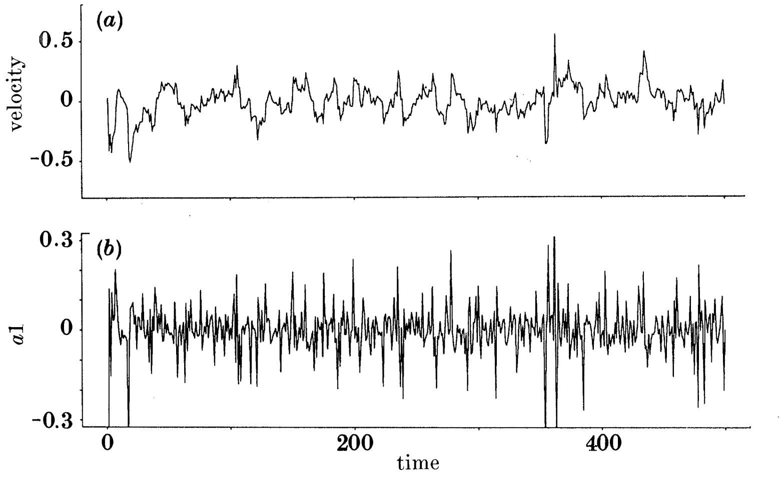

Figure 3. Hyperbolic cosine process with $\rho = 0.8$ (*a*) velocity; (*b*) lag-1 differences.

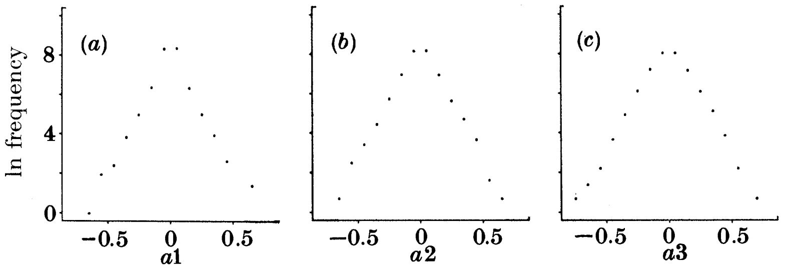

Figure 4. Distribution of differences from process shown in figure 3. (*a*) Lag 1; (*b*) lag 2; (*c*) lag 3.

differencing, as discussed in §5, we are led to compare these empirical velocity and velocity difference processes with those of an autoregressive hyperbolic cosine process with regression coefficient $\frac{1}{2}$, since for that value the lag-1 process will also have hyperbolic cosine marginals. The result of a simulation of this hyperbolic cosine autoregression is shown together with the derived lag-1 difference process in figure 5.

The variation of the lag-1 difference process in figure 5*b* appears quite similar to the observed lag-1 process in figure 1*b*. This is to be expected since we have chosen the value $\rho = 0.5$ to make the tail behaviour of the marginal distribution of the lag-1 process accord with the data. The difference process corresponding to the hyperbolic cosine autoregression with $\rho = 0.8$ (figure 3*b*) shows too many extreme and too many moderate values to be a reasonable model for the observed velocity differences from the Ferring data. On the other hand, the hyperbolic cosine autoregression with $\rho = 0.8$ is more similar to the observed velocity process than that with $\rho = 0.5$. The velocity data show a behaviour with correlation over considerably longer ranges than any of the two autoregressions, the autoregression with the smallest value of ρ being least similar to the observations.

In the light of the discussion above it seems of interest to study the correlation structure in the data more closely. The apparent correlation over long ranges could also be caused by non-stationarity. Physically this is not very plausible, but to investigate this possibility we studied the estimated autocorrelations for the Ferring

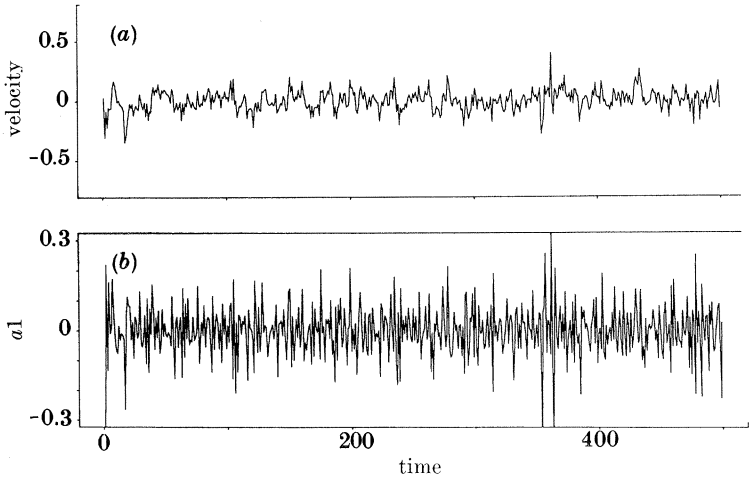

Figure 5. Hyperbolic cosine process with $\rho = 0.5$ (*a*) velocity; (*b*) lag-1 differences.

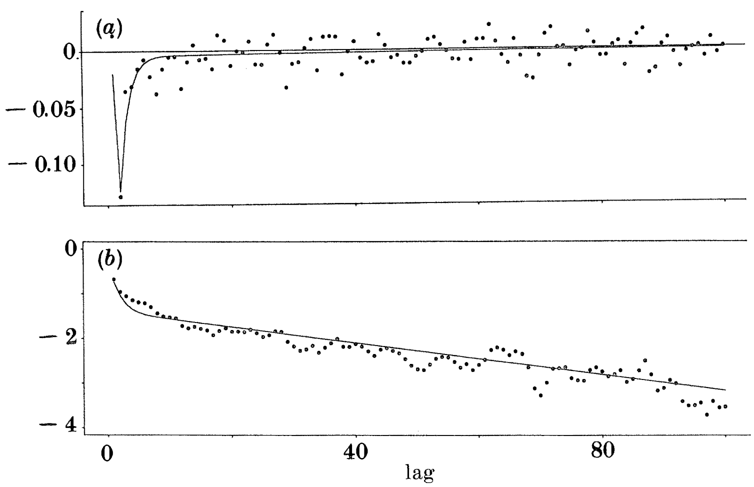

Figure 6. (*a*) Run-1 autocorrelations; (*b*) run-1 transformed autocorrelations.

velocity difference process. The plot of these autocorrelations (figure 6*a*) is not very informative about the correlation over longer ranges. If the velocity process is stationary, then necessarily the sum of the autocorrelations for the difference process from lag 1 and upwards equals -0.5. Therefore, in figure 6*b* we have plotted against k the quantities

$$\ln\left(\frac{1}{2}+\sum_{i=1}^{k}\hat{\rho}_i\right),$$

where $\hat{\rho}_i$ denotes the estimated lag i autocorrelation for the difference process. The curve does not invalidate the assumption about stationarity and shows a linear behaviour for a large range of lag values. If the velocity process is an autoregression of order one, a straight line is expected for all lag values. It therefore appears appropriate to try to model the velocity process as the sum of two independent autoregressions. This type of model was discussed in §5. The process with the largest

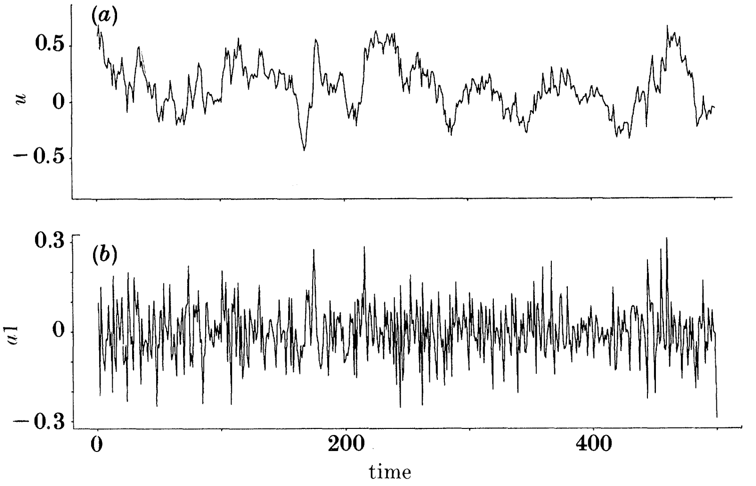

Figure 7. (*a*) Sum of autoregressions, velocity; (*b*) sum of autoregressions, lag-1 differences.

regression coefficient describes the correlations over long ranges, and the other modifies the correlation structure at short ranges. Before we fit a model of this type to the Ferring data, we take a second look at figure 6*a*.

The relatively large negative value of the lag-1 autocorrelation in figure 6*a* can presumably not be accounted for by a sum of two autoregressions. However, as mentioned in §3, the measuring equipment at Ferring averaged over a time-range comparable to the time between observations. This might well have affected the lag-1 autocorrelation. To cope with this problem, we use the following model for the velocity process. At equidistant time points separated by 0.05 s (half the sampling distance in the data) we define a process which is the sum of two independent autoregressions. Let ρ_1 and ρ_2 denote the two regression coefficients, and let χ be the ratio between the variance of the process with regression coefficient ρ_1 and that of the other process. The lag k autocorrelation for this process is given by (5.21). We model the measured velocity process by the process obtained after averaging over two time points. The parameter values $\rho_1 = 0.9912$, $\rho_2 = 0.7$ and $\chi = 15.7$ give autocorrelations that fit the observed ones rather well. The corresponding curves are plotted in figure 6. Note that ρ_1 and ρ_2 should be squared (giving 0.98 and 0.49) to be comparable with the ρ values discussed earlier. A simulation of the model with the estimated parameters is given in figure 7 together with the corresponding lag-1 difference process. For the innovation distribution we used (5.13) with $\rho = 0.7$ for both autoregressions. The simulated sample path appears very similar to the data in figure 1. In conclusion, it seems worthwhile to investigate more closely the modelling of turbulence by means of processes of this type.

10. Discussion: the need for further experimental and theoretical work

In this paper, only some first steps have been taken towards constructing parametric statistical models for turbulence and comparing these with measurements and with the desiderata in §4. The models developed here should be more thoroughly checked against data. We also plan to obtain better observations. In particular, measurements with equipment that averages only over ranges that are small compared to the inertial subrange would be desirable and are feasible.

There are several directions in which to continue the model development. The one-dimensional models discussed in this paper can be improved. In particular, it should be investigated whether the correlation over long ranges is best modelled by a sum of two autoregressions, as discussed in §9, or by a process with long-range dependence in the formal sense.

An interesting problem is how to make the necessary step to a model for the three-dimensional velocity vector. At Ferring all three velocity components were recorded, so there are already data to build on. A complication is that Taylor's frozen field hypothesis (§3), which is used to relate temporal variation to spatial variation, is only applicable to the streamwise velocity component. Note also that Taylor's hypothesis only enables us to get information about streamwise derivatives from a time series of point measurements.

An integrated model describing all scales of the turbulence should be developed and related to such large-scale structures as wind shear. The dynamic equations relating the higher-order moments of the velocity field provide an important criterion for deciding whether a given three-dimensional model makes physical sense. These equations are derived from the Navier–Stokes equations (see Monin & Yaglom 1975, §14).

For comparison with Kolmogorov's refined theory, it would be interesting to study, analytically or by simulation, the distribution of the one-dimensional marginals of the process $(\partial u/\partial t)^2$ for each of the models proposed. Here, as above, u denotes the streamwise velocity component. This process is closely related to the dissipation process, and the one-dimensional marginals of these two processes are usually assumed to be identical apart from a scale transformation. With a model for the spatial variation of the three-dimensional velocity vector it would be possible to study the distribution of the dissipation directly, whereas this can not be done with a one-dimensional temporal model. Therefore, it is so far necessary to go via $(\partial u/\partial t)^2$.

Another direction of research, which we plan to undertake, is the study in their own right of the models proposed in this paper. Here we think in particular about the sum of two autoregressions, the continuous time autoregressions and the processes with long range dependence. For the last class of processes the self-similarity properties are of obvious interest, as also in the turbulence context.

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